A Reproduced Copy

OF

Reproduced for NASA

by the

NASA Scientific and Technical Information Facility

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS
INSTRUMENTATION LABORATORY
CAMBRIDGE, MASS. 02139

C. S. DRAPER
DIRECTOR

69-341L-7
October 30, 1969

Mr. Thomas B. Murtagh
Code FM8
NASA/Manned Spacecraft Center
Advanced Mission Design Branch
Houston, Texas 77058

Dear Mr. Murtagh:

Pursuant to the technical report provisions and the schedule of paragraph 6.1 of contract NAS-9-9024 the third quarterly progress report is hereby submitted. This describes the period of work from 1 August 1969 to 31 October 1969.

Nineteen copies of this report are enclosed. Additional copies have been distributed as indicated below.

Very truly yours,

Donald C Fraser

Dr. Donald C. Fraser
Program Manager

DCF/mee

Enclosures (19)
cc: NASA/Manned Spacecraft Center
J. D. Wilcox/JC-33(2)
J. Funk/FM8(2)
J. T. Wheeler/BM-7(2)

MIT Instrumentation Laboratory

R. H. Battin	G. M. Levine
S. R. Croopnick	E. S. Muller
J. C. Deckert	J. E. Potter
D. G. Hoag	W. M. Robertson
J. H. Laning	Central Files (3)

Letter of Transmittal Only:
D. Dolan
E. Howard

NASA CR108543

Third Quarterly Progress Report, Contract NAS-9-9024

1 August 1969 - 31 October 1969

INTRODUCTION

This quarterly progress report to the technical monitor of the referenced contract is organized into four main sections as follows:

Section I In the second quarterly progress report under this contract, a satellite state vector was defined which consisted of the position deviation and two velocity deviation components, expressed in a rotating coordinate system, plus the deviation from the nominal energy of the satellite. The inclusion of the energy deviation in the state limited the growth rate of the total energy of the satellite, reflecting the fact that the satellite-planet system is conservative. This section presents a technique for determining the scalar measurement geometry vector for the above mentioned state when the corresponding measurement geometry vector for the state vector consisting of all six position and velocity variations, expressed in inertial coordinates, is known. This technique is valid when the scalar measurement is a function of the satellite position only.

Section II In the second quarterly progress report a minimal approximation to the differential equation for the estimation error covariance matrix was developed, assuming no measurements were taken. The approximation has a free parameter associated with it, and the determination of the best time history of this free parameter leads naturally to a two point boundary value optimization problem. In this section, the above approximating technique is extended to include measurements. Arguments are presented which indicate that the measurement incorporation should not be of the straight-forward minimum variance type. A form of measurement incorporation, involving another free parameter, is chosen that allows considerable latitude. The necessary conditions for the optimal choices of the free parameters are then presented, and a new two point boundary value problem results.

Section III Any computer program to solve the two point boundary value problem developed in Section II will need an input matrix Q(t), defined to be the covariance matrix of the disturbance vector arising from gravitational anomalies. This section presents one method of approximating Q, given some model of the gravitational potential of the attracting body. The section also presents an explicit formula for the mean squared value of the disturbing potential averaged over a sphere using a truncated spherical harmonic model of the gravitational potential of the attracting body.

Section IV This section contains Chapter 2 of a doctoral thesis written by Steven Croopnick. The purpose of this thesis is to study and then model various gravitational anomalies, with an end toward better orbit prediction capabilities. Chapter 2 is a development of the gravitational potential of a distributed mass expressed in spherical harmonics derived from basic principles. This chapter is intended as an easy to follow introduction to the classical descriptions of a gravity field, the results of which will be used in later chapters for comparisons with other gravity models. It is felt that this section is fundamental and easily readable, and it has been tested in the classroom.

The following people have contributed to the effort summarized in this Third Quarterly Progress Report:

R. H. Battin

S. R. Croopnick

J. C. Deckert

D. C. Fraser

J. E. Potter

I. Measurement Partials for the Adjoined State Variables ($\underline{r}$, $\underline{v}$)

A. General Development

The purpose of this section is to develop the relationship between the measurement geometry vector $\underline{b}$, associated with the usual state vector

$$\underline{x} = [\, \delta r_1,\ \delta r_2,\ \delta r_3,\ \delta v_1,\ \delta v_2,\ \delta v_3]^T$$

expressed in inertial coordinates and the measurement geometry vector $\underline{b}^*$, associated with the adjoined state vector

$$\underline{x}^* = [\, \delta h\ ,\ \delta r_2',\ \delta r_3',\ \delta r_1',\ \delta v_2',\ \delta v_3']^T$$

expressed in rotating coordinates. The scalar h is the total energy of a satellite in orbit about a rotating planet defined in Reference 1.

The definition of the measurement geometry vector $\underline{d}$ for any state variable $\delta\underline{z}$, the variation in $\underline{z}$, is given by

$$\underline{d}^T = \frac{\partial Q}{\partial \underline{z}}$$

where Q is the scalar quantity being measured, and the convention used is that the derivative of a scalar with respect to a column vector is a row vector. Now, if we wish to find the measurement geometry vector $\underline{c}$ associated with the vector $\delta\underline{z}$ expressed in some rotated coordinate system, which we shall call $\delta\underline{y}$, we proceed as follows. By the definition of the measurement geometry vector, we have

$$\underline{c}^T = \frac{\partial Q}{\partial \underline{y}} = \frac{\partial Q}{\partial \underline{z}} \frac{\partial \underline{z}}{\partial \underline{y}} = \underline{d}^T \frac{\partial \underline{z}}{\partial \underline{y}} \qquad (1)$$

We shall call A the coordinate transformation matrix between $\underline{z}$ and $\underline{y}$, i.e.

$$\underline{y} = A\underline{z} \tag{2}$$

Because A is an orthogonal transformation matrix, $A^{-1} = A^T$ and thus

$$\underline{z} = A^T \underline{y}; \qquad \frac{\partial \underline{z}}{\partial \underline{y}} = A^T \tag{3}$$

Substituting Eq. (3) into Eq. (1) and remembering the definitions of $\underline{c}$ and d, we see that the measurement geometry vectors are also related by A, i.e.

$$\underline{c} = A\underline{d} \tag{4}$$

Now, in order to find $\underline{b}^*$ from $\underline{b}$ we proceed as follows. The measurement geometry vector $\underline{b}$ is defined as

$$\underline{b} = \left[\frac{\partial f}{\partial \underline{r}}, \frac{\partial f}{\partial \underline{v}} \right]^T \tag{5}$$

where $\underline{r}$ and $\underline{v}$ are expressed in inertial coordinates and

$$Q = f(\underline{r}, \underline{v}) \tag{6}$$

We shall assume that the coordinate system of $\underline{b}^*$, the prime system, is related to the inertial coordinate system of $\underline{b}$ as follows

$$\underline{r}' = M\underline{r}, \qquad \underline{v}' = M\underline{v} \tag{7}$$

Utilizing Eqs. (7), (4) and (2) we may define the vector $\underline{b}'$ as

$$\underline{b}' = \begin{bmatrix} M & O \\ O & M \end{bmatrix} \underline{b} = \left[\frac{\partial f'}{\partial \underline{r}'}, \frac{\partial f'}{\partial \underline{v}'} \right]^T \tag{8}$$

where Q has been expressed as the function $f'(\underline{r}', \underline{v}')$. Now we assume that v_1' may be written as

$$v_1' = \alpha(h, \underline{r}', v_2', v_3', u) \tag{9}$$

where u is massless potential energy. Thus, the scalar measurement Q may be written as

$$Q = f'(r_1', r_2', r_3', \alpha, v_2', v_3') \tag{10}$$

This notation means that the function α has been substituted in f' wherever the term v_1' occurred. The measurement vector $\underline{b}^*$ is defined as

$$\underline{b}^* = \left[\frac{\partial Q}{\partial h}, \frac{\partial Q}{\partial r_2'}, \frac{\partial Q}{\partial r_3'}, \frac{\partial Q}{\partial r_1'}, \frac{\partial Q}{\partial v_2'}, \frac{\partial Q}{\partial v_3'} \right]^T \tag{11}$$

The chain rule stipulates that the partial derivatives of Q may be found by the following formula

$$\left.\begin{aligned} \frac{\partial Q}{\partial x} &= \frac{\partial f'}{\partial x} + \frac{\partial f'}{\partial \alpha}\frac{\partial \alpha}{\partial x} & x &= r_1', r_2', r_3', v_2', v_3' \\ &= \frac{\partial f'}{\partial \alpha}\frac{\partial \alpha}{\partial h} & x &= h \end{aligned}\right\} \tag{12}$$

Utilizing Eqs. (8), (11) and (12), $\underline{b}^*$ may be written in component form as

$$\left.\begin{aligned} b_1^* &= b_4' \frac{\partial \alpha}{\partial h} \\ b_2^* &= b_2' + b_4' \frac{\partial \alpha}{\partial r_2'} \\ b_3^* &= b_3' + b_4' \frac{\partial \alpha}{\partial r_3'} \\ b_4^* &= b_1' + b_4' \frac{\partial \alpha}{\partial r_1'} \\ b_5^* &= b_5' + b_4' \frac{\partial \alpha}{\partial v_2'} \\ b_6^* &= b_6' + b_4' \frac{\partial \alpha}{\partial v_3'} \end{aligned}\right\} \tag{13}$$

where all variables assume their nominal values

B. Definition of the Coordinate Transformation Matrix M

In Reference 1, the rotating coordinate system was defined by the unit vectors.

$\underline{u}_1$ along the nominal two body inertial velocity vector $\underline{v}$

$\underline{u}_3$ along $\underline{v} \times \underline{r}$

$\underline{u}_2 = \underline{u}_3 \times \underline{u}_1$ (along + $\underline{r}$ for circular nominal orbit)

Thus the matrix M, used to compute $\underline{b}'$ from $\underline{b}$ in Eq. (8), is given by

$$M = \begin{bmatrix} \underline{u}_1^T \\ \underline{u}_2^T \\ (\underline{u}_1 \times \underline{u}_2)^T \end{bmatrix} \tag{14}$$

where

$\underline{u}_1$ = unit vector along $\underline{v}$

$\underline{u}_2$ = unit vector along $(\underline{r} - (\underline{r} \cdot \underline{v})\, \underline{v}/|\underline{v}|^2)$

$\underline{v}$ = nominal inertial velocity vector expressed in inertial coordinates

$\underline{r}$ = nominal position vector expressed in inertial coordinates.

C. Derivation of the Partial Derivatives

In Reference 1, the relationship between h, $\underline{r}$ and $\underline{v}$ was shown to be

$$h = \frac{1}{2} \underline{v} \cdot \underline{v} - \underline{\omega} \cdot (\underline{r} \times \underline{v}) + u \tag{15}$$

where $\underline{\omega}$ is the angular velocity of the attracting body, u is the massless potential energy, $\underline{v}$ is the inertial velocity, $\underline{r}$ is the position of the satellite, and all vectors are expressed in the rotating coordinate system defined by $\underline{u}_1$, $\underline{u}_2$ and $\underline{u}_3$ above. (Note that from now on, the prime notation is dropped and $\underline{r}$, $\underline{v}$ and $\underline{\omega}$ will be assumed to be expressed in the rotating system.) Expanding Eq. (15) and rearranging gives

$$v_1^2 + \ell\, v_1 + n = 0$$

where

$$\left.\begin{aligned} \ell &= 2\,(r_2\omega_3 - r_3\omega_2) \\ n &= v_2^2 + v_3^2 + 2v_2(r_3\omega_1 - r_1\omega_3) + 2v_3(r_1\omega_2 - r_2\omega_1) + 2u - 2h \end{aligned}\right\} \quad (16)$$

Thus v_1 is given by

$$v_1 = -\ell/2 \pm \sqrt{(\ell^2/4) - n} \quad (17)$$

The partial derivative of v_1 with respect to x is given by

$$\frac{\partial v_1}{\partial x} = -\frac{1}{2}\frac{\partial \ell}{\partial x} \pm \frac{1}{2}\left(\frac{\ell}{2}\frac{\partial \ell}{\partial x} - \frac{\partial n}{\partial x}\right) \Big/ \sqrt{(\ell^2/4) - n} \quad (18)$$

But Eq. (17) may be rearranged to give

$$\pm\sqrt{(\ell^2/4) - n} = v_1 + \ell/2 \quad (19)$$

Substituting Eq. (19) into (18) yields

$$\frac{\partial v_1}{\partial x} = -\frac{1}{2}\frac{\partial \ell}{\partial x} + \left(\frac{\ell}{2}\frac{\partial \ell}{\partial x} - \frac{\partial n}{\partial x}\right) \Big/ 2\,(v_1 + \ell/2) \quad (20)$$

Utilizing Eqs. (16) and (20) we obtain

$$\frac{\partial v_1}{\partial r_1} = (v_2\omega_3 - v_3\omega_2 + g_1) \,/\, (v_1 + r_2\omega_3 - r_3\omega_2)$$

$$\frac{\partial v_1}{\partial r_2} = -\omega_3 + \left[(r_2\omega_3 - r_3\omega_2)\,\omega_3 + v_3\omega_1 + g_2\right] / (v_1 + r_2\omega_3 - r_3\omega_2)$$

$$
\left.
\begin{aligned}
\frac{\partial v_1}{\partial r_3} &= \omega_2 + \Big[(r_2\omega_3 - r_3\omega_2)(-\omega_2) - v_2\omega_1 + g_3 \Big] / (v_1 + r_2\omega_3 - r_3\omega_2) \\
\frac{\partial v_1}{\partial v_2} &= (-v_2 + r_1\omega_3 - r_3\omega_1) \,/\, (v_1 + r_2\omega_3 - r_3\omega_2) \\
\frac{\partial v_1}{\partial v_3} &= (-v_3 + r_2\omega_1 - r_1\omega_2) \,/\, (v_1 + r_2\omega_3 - r_3\omega_2) \\
\frac{\partial v_1}{\partial h} &= 1 \,/\, (v_1 + r_2\omega_3 - r_3\omega_2)
\end{aligned}
\right\} \tag{21}
$$

where $\underline{g}$ is the is the acceleration vector.

Recalling the definition of α given in section A and the fact that we have dropped the prime notation, it is clear that the six expressions in Eqs. (21) are the terms $\frac{\partial\alpha}{\partial r_1}'$, $\frac{\partial\alpha}{\partial r_2}'$, $\frac{\partial\alpha}{\partial r_3}'$, $\frac{\partial\alpha}{\partial v_2}'$, $\frac{\partial\alpha}{\partial v_3}'$ and $\frac{\partial\alpha}{\partial h}$ appearing in Eqs. (13). It should also be remembered that the rotating coordinate system is chosen such that nominally the only nonzero component of the velocity vector is along the "1" direction and $g_3 = r_3 = 0$. Writing the magnitude of the nominal velocity as v, dropping the zero-valued components, and substituting Eqs. (21) into Eqs.(13) gives the components of $\underline{b}^*$ as follows.

$$
\left.
\begin{aligned}
b_1^* &= b_4'/k \\
b_2^* &= b_2' + b_4'(-\omega_3 v + g_2)/k \\
b_3^* &= b_3' + b_4'\,\omega_2 v/k \\
b_4^* &= b_1' + b_4' g_1/k \\
b_5^* &= b_5' + b_4'\, r_1\omega_3 \,/k \\
b_6^* &= b_6' + b_4'(r_2\omega_1 - r_1\omega_2)/k
\end{aligned}
\right\} \tag{22}
$$

where $k = v + r_2\omega_3$ and $\underline{b}'$ is defined by Eq. (8)

D. A Word on Measurements

The quantity δQ, the difference between the expected value of the scalar measurement and the actual measurement, may be written as

$$\delta Q = \underline{b}^{*T} \underline{x}^* + b_4{}'\beta\ \delta u + n$$

where $\beta = \frac{\partial v_1}{\partial u}$, δu is the variation in massless potential energy from the nominal, and n is white noise. The scalar β may be computed from Eq. (20) to be

$$\beta = -\frac{1}{k}$$

Utilizing linear measurement incorporation of the form

$$\hat{\underline{x}}^{*\prime} = \hat{\underline{x}}^* + \underline{w}\ (\delta Q - \underline{b}^{*T} \hat{\underline{x}}^*)$$

results in an estimation error after the measurement of

$$\underline{e}' = (I - \underline{w}\,\underline{b}^{*T})\,\underline{e} + \underline{w}\,\frac{b'_4}{k}\ \delta u + \underline{w}n \qquad (23)$$

where a prime indicates the value of a variable just after the measurement, the carat denotes an estimated quantity, and $\underline{e} = \hat{\underline{x}}^* - \underline{x}^*$.

We note that as long as $b_4{}' = 0$, i.e. as long as the scalar being measured is a function of position only, then Eq. (23) becomes

$$\underline{e}' = (I - \underline{w}\,\underline{b}^{*T})\,\underline{e} + \underline{w}n \qquad (24)$$

and the measurement equations assumed in Section II are valid.

In the next study period, the incorporation of a velocity measurement will be investigated, together with a formulation of the problem using the universal state variables $\underline{r}_o$ and $\underline{v}_o$ presented in Reference 1.

II. Measurement Incorporation

Introduction

In this section, the importance of recognizing the existence of a component of the estimation error uncorrelated with the driving force is demonstrated. A new cost function, to be minimized at each measurement time and having a free scalar parameter μ, is introduced. The form of this new measurement cost function permits the covariance matrices of the correlated and uncorrelated estimation errors to be weighted differently at the measurement times, thus allowing for the differences in form of their differential equations, valid between measurement times. Finally, the necessary conditions for the optimal values of μ and λ, the free parameter introduced in Reference 1, are derived with the cost function being the trace of the product of a symmetric matrix and the covariance matrix at some final time of interest.

A. The Two Uncorrelated Estimation Errors

The differential equation for the state is given by

$$\dot{\underline{x}} = F\underline{x} + \underline{d}$$

where $\underline{d}$ is the driving force due to gravitational anomalies and is correlated with the state.

The differential equation for the state estimate is given by

$$\dot{\hat{\underline{x}}} = F\hat{\underline{x}}$$

Thus the estimation error, $\underline{e} = \hat{\underline{x}} - \underline{x}$, has a covariance $P = \overline{\underline{e}\underline{e}^T}$ whose differential equation is given by

$$\dot{P} = FP + PF^T + \overline{\underline{e}\underline{d}^T} + \overline{\underline{d}\underline{e}^T} \tag{1}$$

where an overbar indicates ensemble average.

It was shown in Reference 1 that the minimal approximation to the term $(\overline{\underline{e}\underline{d}^T} + \overline{\underline{d}\underline{e}^T})$ of Eq. (1) is given by

$$\lambda\overline{\underline{e}\underline{e}^T} + \overline{\underline{d}\underline{d}^T}/\lambda \tag{2}$$

with λ positive and lying between the square roots of the extreme eigenvalues of the matrix $(\overline{\underline{e}\underline{e}^T})^{-1}(\overline{\underline{d}\underline{d}^T})$. In the normal situation when $\overline{\underline{e}\underline{e}^T}$ is of full rank, Eq. (2) is a positive definite approximation to the possibly indefinite quantity $(\overline{\underline{e}\underline{d}^T} + \overline{\underline{d}\underline{e}^T})$, and this is a built in conservatism of the approximation to the cross correlation which must be recognized when implementing the scheme.

Consider for a moment the possibility that the state estimation error $\underline{e}$ may be written as the sum of two uncorrelated components, $\underline{e}_n$ and $\underline{e}_d$, of which only $\underline{e}_d$ is correlated with the driving force $\underline{d}$. In that case Eq. (1) becomes

$$\dot{P} = FP + PF^T + \overline{\underline{e}_d\underline{d}^T} + \overline{\underline{d}\underline{e}_d^T} \tag{3}$$

If we substitute the minimal approximation to the cross correlation into Eq. (3), the differential equation becomes

$$\dot{P} = FP + PF^T + \lambda\overline{\underline{e}_d\underline{e}_d^T} + \overline{\underline{d}\underline{d}^T}/\lambda \tag{4}$$

Now, if $\underline{e}$ is assumed to be entirely correlated with $\underline{d}$, Eqs. (1) and (2) yield

$$\dot{P}' = FP' + P'F^T + \lambda\overline{\underline{e}\underline{e}^T} + \overline{\underline{d}\underline{d}^T}/\lambda \tag{5}$$

where $P'(t_o) = P(t_o)$. Now, the difference between the estimated covariance matrix obtained by integrating Eq. (5) and the estimated covariance matrix obtained by integrating Eq. (4), which we will call D, obeys the differential equation

$$\dot{D} = (\dot{P}' - \dot{P}) = FD + DF^T + \lambda \overline{\underline{e}_n \underline{e}_n^T} \tag{6}$$

if we assume that λ is the same in Eqs. (4) and (5). Using the matrix variation of constants formula and the initial condition $D(t_o) = 0$, Eq. (6) has the solution

$$D(t) = \lambda \int_{t_o}^{t} \Phi(t, \sigma) \overline{\underline{e}_n \underline{e}_n^T} \Phi^T (t, \sigma) \, d\sigma \tag{7}$$

where $\Phi(t, t_o)$ is the state transition matrix defined by the equations

$$\dot{\underline{x}} = F\underline{x} \quad , \quad \underline{x}(t) = \Phi(t, t_o) \underline{x}(t_o)$$

Since λ must be positive, it is clear that D(t) is at least a positive semidefinite matrix which cannot decrease in time.

To recapitulate, it has been demonstrated that if indeed the estimation error $\underline{e}$ has a component $\underline{e}_n$ which is uncorrelated with the driving force $\underline{d}$, then the use in the differential equation for the covariance matrix of the minimal approximation to the cross correlation $\overline{\underline{e}\underline{d}^T + \underline{d}\underline{e}^T}$ results in an estimated covariance matrix which is conservative, above and beyond the built in conservatism inherent in the approximation, by the matrix D(t). Thus it would seem wise to recognize any component of $\underline{e}$ uncorrelated with $\underline{d}$ and use Eq. (4) instead of Eq. (5) to approximate the differential equation of the covariance matrix.

It will be assumed here that any measurements taken will be corrupted by white noise, which is uncorrelated with the state. Since any linear measurement incorporation scheme will premultiply the existing error vector by a matrix and add a term due to white noise, and since the present error is due to the effects of previous uncorrelated measurement errors plus the effects of the errors correlated with the driving force $\underline{d}$, it seems reasonable at this time to postulate the existence of two uncorrelated components of the state estimation error $\underline{e}$, of which only one component is correlated with $\underline{d}$. It will later be shown that this is indeed the case.

Now we make the following definitions

$$\underline{e} = \underline{e}_d + \underline{e}_n, \ \overline{\underline{e}_d\underline{e}_n{}^T} = 0, \ \overline{\underline{e}_n\underline{d}^T} = 0, \ P_d = \overline{\underline{e}_d\underline{e}_d{}^T}, \ P_n = \overline{\underline{e}_n\underline{e}_n{}^T} \tag{8}$$

Consequently, there follows

$$P = P_n + P_d \tag{9}$$

Substitution of Eqs. (8) and (9) into (1) yields

$$\dot{P}_n + \dot{P}_d = F(P_n + P_d) + (P_n + P_d)F^T + \overline{\underline{e}_d\underline{d}^T} + \overline{\underline{d}\underline{e}_d{}^T}$$

We shall rewrite the above equation as the sum of the following two equations, which must hold if $\underline{e}_n$ and $\underline{e}_d$ are uncorrelated.

$$\left.\begin{aligned} \dot{P}_n &= FP_n + P_nF^T \\ \dot{P}_d &= FP_d + P_dF^T + \overline{\underline{e}_d\underline{d}^T} + \overline{\underline{d}\underline{e}_d{}^T} \end{aligned}\right\} \tag{10}$$

We note that the differential equations for P_n and P_d, Eqs. (10), are completely uncoupled.

Because of its simplicity of form and ease of implementation, we will assume linear measurement incorporation. We shall now proceed to demonstrate how linear measurement incorporation alters the values of $\underline{e}_n$ and $\underline{e}_d$ and keeps them uncorrelated.

Assume a scalar measurement m is taken, where m is given by

$$m = \underline{b}^T \underline{x} + n \tag{11}$$

with n white noise. Assume that the measurement is incorporated into the state estimate with a linear filter of the form

$$\hat{\underline{x}}' = \hat{\underline{x}} + \underline{w}(m - \underline{b}^T \hat{\underline{x}}) \tag{12}$$

where a prime denotes the value of the estimate just after measurement incorporation and the absence of a prime denotes the value of that estimate just prior to measurement incorporation.

Substitution of Eq (11) into Eq. (12) together with the definition of $\underline{e}$ yields

$$\underline{e}' = (I - \underline{w}\underline{b}^T)\underline{e} + \underline{w}n$$

where I is the identity matrix. This equation may be written as the sum of the following two equations

$$\left.\begin{aligned} \underline{e}'_n &= (I - \underline{w}\underline{b}^T)\underline{e}_n + \underline{w}n \\ \underline{e}'_d &= (I - \underline{w}\underline{b}^T)\underline{e}_d \end{aligned}\right\} \tag{13}$$

Using the definition of $\underline{e}$, the differential equation for the state, and Eq. (8), we may write $\underline{e}_n$ and $\underline{e}_d$ at the time t as

$$\left.\begin{aligned} \underline{e}_n(t) &= \Phi(t, t_o)\underline{e}_n(t_o) \\ \underline{e}_d(t) &= \Phi(t, t_o)\underline{e}_d(t_o) + \int_{t_o}^{t} \Phi(t, \sigma)\underline{d}(\sigma)\, d\sigma \end{aligned}\right\} \tag{13a}$$

where $\Phi(t, t_o)$ is the state transition matrix for $\dot{\underline{x}} = F\underline{x}$
From Eqs. (13a) we see that $\underline{e}_n(t)$ is uncorrelated with $\underline{d}$ if $\underline{e}_n(t_o)$ is uncorrelated with $\underline{d}$, and likewise Eqs. (13a) and (13) demonstrate that $\underline{e}_n'(t)$ is uncorrelated with $\underline{d}$ if $\underline{e}_n(t_o)$ is uncorrelated with $\underline{d}$,

since n is white noise. Since $\underline{e}_n(t_o)$ is uncorrelated with $\underline{d}$ by definition, we see that $\underline{e}_n(t)$ is uncorrelated with $\underline{d}$ for all times $t \geq t_o$. (Although the $\underline{w}_n$ term in Eqs. (13) could have been added to either the $\underline{e}'_n$ or $\underline{e}_d'$ equation without altering their correlation properties, it is added to $\underline{e}_n'$ equation in order to attribute all of the uncorrelated error to $\underline{e}_n$, the virtue of which was demonstrated earlier.)

Using Eqs. (13) and (9) the behaviors of P_n and P_d after measurement incorporation are given by:

$$\left.\begin{aligned} P_n' &= (I - \underline{w}\underline{b}^T)\, P_n\, (I - \underline{b}\underline{w}^T) + \underline{w}\underline{w}^T\, \overline{n^2} \\ P_d' &= (I - \underline{w}\underline{b}^T)\, P_d\, (I - \underline{b}\underline{w}^T) \end{aligned}\right\} \qquad (14)$$

Eqs. (14) demonstrate that if the measurement weighting vector $\underline{w}$ is non-zero, P_n will be non-zero after the first measurement incorporation.

In review, Eqs. (10) indicate the behavior of the covariance matrix of uncorrelated error, P_n, and the covariance matrix of the correlated error, P_d, between measurements. However, the cross correlation terms in the $\dot{P}_d$ equation are unknown and must be replaced by the minimal approximation derived in Reference 1. This results in the following set of equations, of which the $\dot{P}_n$ equation is exact and the $\dot{P}_d$ equation is approximate

$$\left.\begin{aligned} \dot{P}_n &= FP_n + P_nF^T \\ \dot{P}_d &= FP_d + P_dF^T + \lambda P_d + \overline{\underline{d}\underline{d}^T}/\lambda \end{aligned}\right\} \qquad (15)$$

Eqs. (14) indicate the changes in P_n and P_d due to a measurement, and, assuming non-zero $\underline{w}$, also demonstrate that P_n will be non-zero after the first measurement regardless of its existence prior to the measurement. The scalar λ in Eq. (15) is a free parameter which must be chosen to suit the particular situation at hand. Any positive

λ lying between the square roots of the extreme eigenvalues of $P_d^{-1} \underline{d}\underline{d}^T$ is admissible.

B. The Optimal Linear Filter

This section deals with the selection of the optimum weighting vector $\underline{w}$ that appears in Eqs. (14) in order to minimize the trace of a symmetric matrix times the estimated covariance matrix at the final time of interest T. The cost J is thus given by.

$$J = tr \left[L\, P(T) \right] = tr \left[L\, (P_d(T) + P_n(T)) \right]$$

(We note that Reference 1 stated that any linear combination of the elements of a symmetric matrix P may be expressed as tr (LP), with L symmetric, and thus the assumption of a symmetric L does not result in a loss of generality.) In the usual situation when there is no driving force $\underline{d}$, the measurement weighting vector $\underline{w}$ is chosen so as to minimize the trace of L times the covariance matrix at the time of the measurement. That is, a new cost function J_k is defined at each measurement time t_k such that

$$J_k = tr \left[L\, (P_n(t_k) + P_d(t_k)) \right]$$

and $\underline{w}(t_k)$ is chosen such that J_k is minimized. Now, looking at Eqs. (15) we see obvious differences between the differential equations for P_d and P_n between measurements. Indeed, since our ultimate goal is to minimize the trace of L times the covariance matrix at the final time T, and since the two components of the covariance matrix, P_n and P_d, obey dissimilar differential equations, it would not seem that P_d and P_n should necessarily be treated equally in the intermediate cost functions J_k. For this reason we propose that at the measurement time t_k, $\underline{w}$ be chosen to minimize the trace of L times a weighted covariance matrix $\widetilde{P}$ defined by

$$\widetilde{P}(t_k) = \mu_k \; P_d(t_k) + P_n(t_k) \qquad (16)$$

(It might well be questioned whether all this is really necessary; that is, can we really know for certain that μ_k is not equal to unity. The answer to this question must be deferred until the necessary conditions for optimality are derived. At that time it will be seen by inspection that the conditions are not identically satisfied for $\mu_k = 1$.)

We will assume an intermediate cost function to be minimized of the form

$$J_k = \mathrm{tr}\,[L\,\widetilde{P}\,(t_k)] \tag{17}$$

This form of intermediate cost function is chosen to allow us to find the weighting vector $\underline{w}_k$ as a function of μ_k in order that the flexibility be present to allow P_n and P_d to be weighted differently in importance at each measurement time. In Subsection C, the necessary conditions will be developed to allow the μ_k's to be chosen so as to minimize the cost $J = \mathrm{tr}\,(L\,P(T))$. We shall now proceed to find the optimal $\underline{w}_k$ as a function of μ_k. Dropping the k subscripts and utilizing Eqs. (16), (17), and (14) we have at the time t_k'

$$J = \mathrm{tr}\,L\widetilde{P} = \mathrm{tr}\left\{L[(I - \underline{wb}^T)\,\widetilde{P}\,(I - \underline{bw}^T) + \underline{ww}^T\,\overline{n^2}]\right\}$$

Taking the variation in J due to a variation in $\underline{w}$ gives

$$\delta J = 2\,\mathrm{tr}\left\{\delta\underline{w}[-\underline{b}^T\widetilde{P}\,(I - \underline{bw}^T) + \underline{w}^T\,\overline{n^2}]\,L\right\}$$

By requiring that δJ be zero for all values of $\delta\underline{w}$, we find that the optimal $\underline{w}$ is given by

$$\begin{aligned}\underline{w} &= \widetilde{P}\,\underline{b}/(\underline{b}^T\widetilde{P}\,\underline{b} + n^2)\\ &= (\mu\,P_d + P_n)\,\underline{b}/[\underline{b}^T(\mu P_d + P_n)\underline{b} + \overline{n^2}]\end{aligned} \tag{18}$$

It should be pointed out that although it cannot be shown that the form of the intermediate cost function given by Eq. (17) is optimal, it is felt that the latitude allowed by the free parameter μ will permit satisfactory results for the minimization of the cost $J = \mathrm{tr}\,(L\,P(T))$.

C. Optimization with Measurements

This section contains a derivation of the necessary conditions for the optimal time history of λ in Eq. (15) and the optimal value of μ in Eq. (18) for each measurement time t_k in order to minimize the trace of L times the covariance matrix at the final time of interest T.

We wish to minimize the scalar cost

$$J = \mathrm{tr}\left\{L\,(P_d(T) + P_n(T))\right\}$$

P_n and P_d are found between measurements by integrating the following differential equations:

$$\dot{P}_n = FP_n + P_nF^T$$

$$\dot{P}_d = FP_d + P_dF^T + \lambda P_d + Q/\lambda$$

where $Q = \overline{\underline{d}\underline{d}^T}$. There are ℓ scalar measurements taken during the mission. At each measurement time t_k, P_d and P_n are altered as follows:

$$P_n(t_k+) = (I - \underline{w}_k\underline{b}_k^T)\,P_n(t_k-)\,(I - \underline{b}_k\underline{w}_k^T) + \underline{w}_k\underline{w}_k^T\,q_k \qquad (19)$$

$$P_d(t_k+) = (I - \underline{w}_k\underline{b}_k^T)\,P_d(t_k-)\,(I - \underline{b}_k\underline{w}_k^T) \qquad (20)$$

where $\underline{w}_k = (\mu_k\,P_d(t_k-) + P_n(t_k-))\,\underline{b}_k/(\underline{b}_k^T(\mu_k\,P_d(t_k-) + P_n(t_k-))\,\underline{b}_k + q_k)$, μ_k is constrained to be non-negative, and $\underline{b}_k$ and q_k are known.

Now we rewrite the cost by adjoining the constraints as follows:

$$J = tr\left\{ L\,(P_d + P_n)\,(T) + \sum_{k=1}^{\ell} (N_k[\,M_k - P_n(t_k+)] + D_k\,[R_k - P_d(t_k+)]\,)\right.$$

$$\left. + \int_{t_o}^{t_1^-} G\,dt + \sum_{k=1}^{\ell-1} \left(\int_{t_k^+}^{t_{k+1}^-} G dt\right) + \int_{t_\ell^+}^{T} G dt \right.$$

where

$$G = C_n\,[FP_n + P_nF^T - \dot{P}_n] + C_d\,[FP_d + P_dF^T + \lambda P_d + Q/\lambda - \dot{P}_d] \tag{21}$$

and D_k, N_k, C_n, and C_d are unknown multipliers to be chosen later, M_k is the right hand side of Eq. (19), and R_k is the right hand side of Eq. (20).

The integration of the last three terms of Eq. (21) by parts results in the following equation for J:

$$J = tr\left\{ L\,(P_d + P_n)\,(T) + \sum_{k=1}^{\ell} (N_k[\,M_k - P_n(t_k+)] + D_k\,[R_k - P_d(t_k+)]\,)\right.$$

$$- (C_nP_n + C_dP_d)\Big|_{t_o}^{t_1^-} \cdot \sum_{k=1}^{\ell-1} (C_nP_n + C_dP_d)\Big|_{t_k^+}^{t_{k+1}^-} - (C_nP_n + C_dP_d)\Big|_{t_\ell^+}^{T}$$

$$+ \int_{t_o}^{t_1^-} H\,dt + \sum_{k=1}^{\ell-1} \left(\int_{t_k^+}^{t_{k+1}^-} H\,dt\right) + \int_{t_\ell^+}^{T} H\,dt$$

where

$$H = [\,\dot{C}_nP_n + C_nFP_n + C_nP_nF^T] + [\,\dot{C}_dP_d + C_dFP_d + C_dP_dF^T$$

$$+ C_d\lambda P_d + C_dQ/\lambda\,] \tag{22}$$

The matrix minimum principle, introduced in Reference 2, states that the necessary conditions for a stationary point of J (i. e. a maximum, minimum, or saddle point) may be found by equating to zero the gradient matrices of J with respect to the control variables and all explicitly stated forms of the state variables that are not constrained to be some a priori value.

A gradient matrix is defined as follows: We are given the scalar function f (X), which is a function of the elements x_{ij} of the matrix X. Then the gradient matrix of f, denoted by $\frac{\partial f}{\partial X}$ is a matrix whose ij th element is given by

$$\left[\frac{\partial f}{\partial X}\right]_{ij} = \frac{\partial f}{\partial x_{ij}}$$

In order to determine the gradient matrices, the following equation, derived in Reference 3, will be used:

$$\frac{\partial}{\partial X} \operatorname{tr} \left(AX\right) = A^T, \quad \text{where A and X are square matrices}$$

Another useful equation is the following:

$$\frac{\partial}{\partial X} \underline{b}^T X \underline{b} = \underline{b}\underline{b}^T$$

The following identities will also be employed:

$$\operatorname{tr} (AB) = \operatorname{tr} (AB)^T = \operatorname{tr} (B^T A^T)$$

$$\operatorname{tr} (AB) = \operatorname{tr} (BA) \qquad \text{if A and B are conformable}$$

Since integration and the trace function are linear operators, the following relation holds:

$$\operatorname{tr} \left[\int A dt\right] = \int \left[\operatorname{tr} A\right] dt$$

Using the equations above, we proceed to take the gradient matrices of

J with respect to P_n, P_d, $P_n(t_k+)$, $P_d(t_k+)$, $P_n(T)$, $P_d(T)$. By equating these gradient matrices to the zero matrix, there will follow necessary conditions on the previously undefined matrix multipliers N_k, D_k, C_n and C_d.

$$\frac{\partial J}{\partial P_n} = \int_{t_o}^{t_1^-} (\dot{C}_n^{\,T} + F^T C_n^{\,T} + C_n^{\,T} F)\, dt + \sum_{k=1}^{\ell-1} \int_{t_k^+}^{t_{k+1}^-} (\dot{C}_n^{\,T} + F^T C_n^{\,T} + C_n^{\,T} F)\, dt$$

$$+ \int_{t_\ell^+}^{T} (\dot{C}_n^{\,T} + F^T C_n^{\,T} + D_n^{\,T} F)\, dt \qquad (23)$$

Setting Eq. (23) to the zero matrix gives the necessary condition that at all but the measurement times

$$\dot{C}_n = -C_n F - F^T C_n \qquad (24)$$

In a similar manner, setting $\frac{\partial J}{\partial P_d}$ to zero yields the condition that at all but the measurement times

$$\dot{C}_d = -C_d F - F^T C_d - \lambda C_d \qquad (25)$$

$$\frac{\partial J}{\partial P_n(t_k+)} = -N_k^{\,T} + C_n^{\,T}(t_k+)$$

Setting the above equation to zero gives the identity

$$N_k = C_n(t_k+) \qquad 1 \le k \le \ell \qquad (26)$$

In a similar manner, setting $\frac{\partial J}{\partial P_n(t_k+)}$ to zero gives

$$D_k = C_d(t_k+) \qquad 1 \le k \le \ell \qquad (27)$$

$$\frac{\partial J}{\partial P_n(T)} = L^T(T) - C_n^T(T)$$

Setting the above equation to zero yields the boundary condition

$$C_n(T) = L(T) \tag{28}$$

Likewise equating $\frac{\partial J}{\partial P_d(T)}$ to zero gives

$$C_d(T) = L(T) \tag{29}$$

At this point we shall take the gradient matrices of J with respect to $P_n(t_k-)$ and $P_d(t_k-)$ and in so doing derive the behavior of C_d and C_n at the times t_k. Utilizing Eqs. (22), (26), and (27) gives

$$\frac{\partial J}{\partial P_n(t_k-)} = \left\{ \frac{\partial}{\partial P_n(t_k-)} [\mathrm{tr}\ (C_n(t_k+)M_k + C_d(t_k+)R_k)] - C_n^T(t_k-) \right\}$$

$$\frac{\partial J}{\partial P_d(t_k-)} = \left\{ \frac{\partial}{\partial P_d(t_k-)} [\mathrm{tr}\ (C_n(t_k+)M_k + C_d(t_k+)R_k)] - C_d^T(t_k-) \right\}$$

Equating each of the above expressions to zero gives

$$C_n^T(t_k-) = \frac{\partial}{\partial P_n(t_k-)} [\mathrm{tr}\ (C_n(t_k+)M_k + C_d(t_k+)R_k)] \tag{30}$$

$$C_d^T(t_k-) = \frac{\partial}{\partial P_d(t_k-)} [\mathrm{tr}\ (C_n(t_k+)M_k + C_d(t_k+)R_k)] \tag{31}$$

Recall that M_k is the right hand side of Eq. (19) and R_k is the right hand side of Eq. (20). Examining those two equations, we realize that the task ahead is tedious, but not impossible. Utilizing the techniques demonstrated by Athans and Schweppe, the right hand sides of Eqs. (30) and (31) have been evaluated. The intermediate expressions are impractical to enumerate, and only the final results are presented. They are as follows:

$$C_n(k-) = A_k^T C_n(k+)A_k + S_k + S_k^T \tag{32}$$

$$C_d(k-) = A_k^T C_d(k+)A_k + \mu_k(S_k + S_k^T) \qquad (33)$$

where

$C_d(k) = C_d(t_k-)$, $C_d(k+) = C_d(t_k+)$, etc.

$A_k = I - \underline{w}_k \underline{b}_k^T$, $B_k = \underline{b}_k \underline{b}_k^T$, $\alpha_k = \underline{b}_k^T(\mu_k P_d(k-) + P_n(k-))\, \underline{b}_k + q_k$

q_k = mean-squared noise for the k^{th} measurement, $\overline{n^2}_k$

$$S_k = \Big\{ -[A_k^T C_d(k+)A_k P_d(k-) + A_k^T C_n(k+)A_k P_n(k-)]\, B_k/\alpha_k + A_k^T C_n(k+)\underline{w}_k \underline{b}_k^T q_k/\alpha_k \Big\}$$

C_n and C_d are symmetric

All that remains now is to set to zero the gradient matrices of J with respect to the control variables $\lambda(t)$ and μ_k.

$$\frac{\partial J}{\partial \lambda} = \mathrm{tr}\left[\int_{t_o}^{t_1^-} (C_d P_d - C_d Q/\lambda^2)\, dt + \sum_{k=1}^{\ell-1} \left(\int_{t_k}^{t_{k+1}^-} (C_d P_d - C_d Q/\lambda^2)\, dt \right) + \int_{t_\ell^+}^{T} (C_d P_d - C_d Q/\lambda^2)\, dt \right]$$

Equating $\frac{\partial J}{\partial \lambda}$ to zero gives the condition that along the optimal trajectory at all but the measurement times and for λ in the admissible region:

$$\mathrm{tr}\,[C_d P_d - C_d Q/\lambda^2] = 0 \qquad (34)$$

Similarly, setting $\frac{\partial J}{\partial \mu_k}$ to zero gives the condition that along the optimal path at the measurement times for μ_k nonnegative:

$$\frac{\partial J}{\partial \mu_k} = 2 \operatorname{tr} \left[S_k P_d(k-) \right] = 0 \tag{35}$$

where

$$C_n(k+) = C_n(t_k+),\ P_n(k-) = P_n(t_k-),\ \text{etc.};\ S_k \text{ as before}$$

In summary, the solution to the problem of choosing $\lambda(t)$ and μ_k in order to minimize the trace of $LP(T)$ leads to a two point boundary value problem. The value of the symmetric costate matrices C_d and C_n are stipulated at the final time by Eqs. (28) and (29) while the values of P_n and P_d at t_o are known. Between measurements the state variables P_n and P_d obey the differential equations given by Eqs. (15). In these same intervals C_d and C_n obey Eqs. (24) and (25). At the measurement times P_n, P_d, C_n and C_d are changed according to Eqs. (19), (20), (32) and (33) respectively. Along the optimal trajectory and for $\lambda(t)$ and μ_k within their admissible bounds Eqs. (34) and (35) are valid.

In answer to the question posed in Section II-B concerning the possibility that μ_k might be identically one, we make the following observation

$$\begin{aligned}\left.\frac{\partial J}{\partial \mu_k}\right|_{\mu_k = 1} = {} & 2\underline{b}_k^T \Big\{ -\alpha_k^2 P_d(k-)C_d(k+)P_d(k-) - 2\,\alpha_k\beta_k P(k-)C_d(k+)P_d(k-) \\ & -\beta_k^2 P(k-)C_d(k+)P(k-) - \alpha_k^2 P_d(k-)C_n(k+)P_n(k-) \\ & -2\alpha_k\beta_k P(k-)C_n(k+)P_n(k-) - \beta_k\gamma_k P(k-)C_n(k+)P(k-) \\ & +q_k \left[\alpha_k P_d(k-)C_n(k+)P(k-) - \beta_k P(k-)C_n(k-)P(k-) \right\} \underline{b}_k / \alpha_k^3 \end{aligned} \tag{36}$$

where $\alpha_k = \underline{b}_k^T P(k-)\underline{b}_k + q_k$, $P(k-) = P_d(k-) + P_n(k-)$

$\beta_k = \underline{b}_k^T P_d(k-)\underline{b}_k$, $\gamma_k = \underline{b}_k^T P_n(k-)\underline{b}_k$

Examination of Eq. (36) would seem to indicate that it is not identically zero, and thus that μ_k is not identically equal to one.

In future studies a computer program utilizing the above derived necessary conditions will be written and solutions from the program investigated. Results will also be presented of an effort to derive the necessary conditions for optimality when the measurements taken are of the general vector type.

III. Mean Square Behavior of the Perturbing Potential of a Planet

In Section IIC of this report, the necessary conditions for the optimality of the approximated estimation error covariance matrix are presented. The matrix Q, defined to be the covariance matrix of the driving noise $\underline{d}$, is an input function of time which will be needed by any optimization program.

There are several methods of approximating Q, based on the assumption that there exists a model of the gravitational field of the attracting body. One particular method of approximation is as follows.

1) Compute the matrix Q', the deterministic matrix $\underline{d}\underline{d}^T$, where $\underline{d}$ is determined from the model of the gravitational field of the attracting body.
2) Average Q'over a sphere of radius r and call it $\overline{Q'}$
3) Approximate Q(t) by $K(t)\,\overline{Q'}\,K^T(t)$, where K(t) is a scale factor matrix which accounts for the difference between the nominal radius r(t) and the radius of the sphere over which $\overline{Q'}$ was averaged. In practice, K(t) would probably be computed on the basis of the radial dependence of the single dominating term in the gravitational model.

This section presents a method for deriving the element of the $\overline{Q'}$ matrix corresponding to the mean squared value of the disturbing potential, i.e.the difference between the actual potential as predicted by a spherical harmonic model and the point mass potential, averaged over a sphere of radius r.

The perturbing potential u at a point may be expressed exactly as

$$u = \frac{\mu}{r}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n\sum_{m=0}^{n}\left\{C_{nm}\cos(m\phi) + S_{nm}\sin(m\phi)\right\}P_{nm}(\cos\theta) \qquad (1)$$

where ϕ = longitude of point

θ = colatitude of point

r = radius of point

a = equatorial radius of planet

μ = product of the gravitation constant and mass of the planet

and $P_{nm}(x)$ are the associated Legendre functions (see Section IV.)

Eq. (1) is the standard expansion of the perturbing potential in spherical harmonics. In all practical applications, the value of n in the expansion is truncated at some upper limit N. It is our purpose to determine $\overline{u^2}$ given C_{nm} and S_{nm} for the truncated harmonic expansion, where the averaging is done over a sphere of radius r, i.e.

$$\overline{u^2} = \frac{1}{A} \iint u^2 \, dA \tag{2}$$

where the integrals are taken over a sphere of radius a and A is the surface area of the sphere.

Substitution of Eq. (1) into Eq. (2) with n truncated at N gives

$$\overline{u^2} = \left(\frac{1}{4\pi r^2}\right) \frac{\mu^2}{r^2} \sum_{n=2}^{N} \left(\frac{a}{r}\right)^{2n} \sum_{m=0}^{n} \left\{ C_{nm}^2 \int_{-\pi}^{\pi} \int_{0}^{\pi} \cos^2(m\phi) \right.$$

$$P_{nm}^2(\cos\theta)\, r^2 \sin\theta \, d\theta \, d\phi + S_{nm}^2 \int_{-\pi}^{\pi} \int_{0}^{\pi} \sin^2(m\phi)$$

$$\left. P_{nm}^2(\cos\theta)\, r^2 \sin\theta \, d\theta \, d\phi \right\} \tag{3}$$

Using the substitution $x = \cos\theta$ and integrating Eq. (3) with respect to ϕ gives

$$\overline{u^2} = \frac{\mu^2}{4\pi r^2} \sum_{n=2}^{N} \left(\frac{a}{r}\right)^{2n} \sum_{m=0}^{n} \left\{ C_{nm}^2 \begin{bmatrix} 2 \text{ if } m=0 \\ 1 \text{ if } m \neq 0 \end{bmatrix} \pi \int_{-1}^{1} P_{nm}^2 (x)dx \right.$$

$$\left. + S_{nm}^2 \begin{bmatrix} 0 \text{ if } m=0 \\ 1 \text{ if } m \neq 0 \end{bmatrix} \pi \int_{-1}^{1} P_{nm}^2 (x)dx \right\} \tag{4}$$

In Reference 4, page 125, it is shown that

$$\int_{-1}^{1} P_{nm}^2 (x)dx = \left(\frac{2}{2n+1}\right) \frac{(n+m)!}{(n-m)!} \tag{5}$$

Substituting Eq. (5) into Eq. (4) gives

$$\overline{u^2} = \frac{\mu^2}{4r^2} \sum_{n=2}^{N} \left(\frac{a}{r}\right)^{2n} \left\{ 2C_{no}^2 \left(\frac{2}{2n+1}\right) \right.$$

$$\left. + (C_{nm}^2 + S_{nm}^2) \frac{(n+m)!}{(n-m)!} \left(\frac{2}{2n+1}\right) \right\} \tag{6}$$

The terms C_{nm} and S_{nm} in Eq.(6) are the so-called unnormalized coefficients. The normalized coefficients, $\bar{C}_{nm}$ and $\bar{S}_{nm}$, are related to C_{nm} and S_{nm} by

$$\begin{bmatrix} C_{nm} \\ S_{nm} \end{bmatrix} = \begin{bmatrix} \bar{C}_{nm} \\ \bar{S}_{nm} \end{bmatrix} \sqrt{\frac{2(2n+1)\ (n-m)!}{(n+m)!}} \qquad m \neq 0 \tag{7}$$

Now, it turns out that for the particular harmonic expansion of interest, provided by Dr. J. H. Laning of the M. I. T. Instrumentation Laboratory, the coefficients for m = 0 are given in the unnormalized form while all the other coefficients are normalized. Substituting

Eq. (7) into Eq. (6) for $m \neq 0$ gives

$$\overline{u^2} = \frac{\mu^2}{r^2} \sum_{n=2}^{N} \left(\frac{a}{r}\right)^{2n} \left\{ \frac{C_{no}^{\ 2}}{2n+1} + \sum_{m=1}^{n} \left(\bar{C}_{nm}^{\ 2} + \bar{S}_{nm}^{\ 2}\right) \right\} \qquad (8)$$

The expansion mentioned above is a model of the earth with the following values for N, μ, and a

$$N = 10$$

$$\mu = 1.407646 \times 10^{16} \ ft^3/sec^2$$

$$a = 20,925,722 \quad ft$$

Eq. (8), evaluated with N, μ and a as given above and for $r = a + 500,000$ ft gave the following result

$$\overline{u^2} = 7.11 \times 10^6 \ ft^4/sec^4$$

References for Sections I - III

1) Second Quarterly Progress Report, Contract NAS-9-9024 M. I. T. Instrumentation Laboratory, Cambridge, Mass., July 28, 1969.

2) Athans, M., "The Matrix Minimum Principle", Information and Control, Volume II, Nos. 5 and 6, November-December 1967.

3) Athans, M. and Schweppe, F. C., "Gradient Matrices and Matrix Calculations, "M. I. T. Lincoln Laboratory, Lexington, Mass., Tech Note 1965 - 53, November 1965.

4) Jackson, D., Fourier Series and Orthogonal Polynominals, Carus Mathematical Monograph No. 6, Buffalo, N. Y., Mathematical Society of America, University of Buffalo, 1941.

IV The Gravitational Potential of a Distributed Mass Expressed in Spherical Harmonics

This section of this quarterly progress report contains Chapter 2 of a doctoral thesis written by Steven Croopnick (hence the section and equation numbers do not correspond to those in the previous sections). The purpose of this thesis is to study and then model various gravitational anomalies, with an end toward better orbit prediction capabilities. Chapter 2 is a development of the gravitational potential of a distributed mass expressed in spherical harmonics derived from basic principles. This Chapter is intended as an easy to follow introduction to the classical decriptions of a gravity field, the results of which will be used in later Chapters for comparisons with other gravity models. It is felt that this section is fundamental and easily readable and has been tested in the classroom.

The research effort of the thesis, has produced computer programs which integrate orbits around the earth using both the classical models and the masspoint and dipole concepts. The resulting orbits were then compared for various combinations of orbital and model conditions to evaluate the effectiveness of the models. Interesting questions and relations between components of the resulting residuals have arisen and are presently being investigated, so that the results of this effort will appear in a later report.

2.1 The Gravitational Potential of a Distributed Mass Expressed in Spherical Harmonics

Let the density of a distributed mass in a region V be represented by $\rho(\underline{r}')$. The gravitational potential $\varphi(\underline{r})$ at a point $\underline{r}$ outside of V due to the mass distribution $\rho(\underline{r})$ is:

$$\varphi(\underline{r}) = G \int_V \frac{dm}{|\underline{r} - \underline{r}'|} = G \int_V \frac{\rho(\underline{r}')}{|\underline{r} - \underline{r}'|} \, dV \qquad (2.1)$$

If γ is the angle between $\underline{r}$ and $\underline{r}'$ (see Fig. 2.1), then,

$$\frac{1}{|\underline{r} - \underline{r}'|} = [\, r^2 + r'^2 - 2rr'\cos\gamma\,]^{-1/2} \qquad (2.2)$$

$$= \frac{1}{r}\left[\, 1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\gamma\,\right]^{-1/2} \qquad (2.3)$$

This function may be expanded as a power series in $\left(\frac{r'}{r}\right)$ as follows:

$$\frac{r}{|\underline{r} - \underline{r}'|} = 1 + \frac{r'}{r}\cos\gamma + \frac{1}{\gamma}\left(\frac{r'}{r}\right)^2 (3\cos^2\gamma - 1) + \ldots \qquad (2.4)$$

$$= \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\gamma) \qquad (2.5)$$

where the P_n's are called Legendre polynomials, and may be calculated according to

$$P_n(\nu) = \frac{1}{2^n n!} \frac{d^n}{d\nu^n} (\nu^2 - 1)^n \qquad (2.6)$$

The potential $\varphi(\underline{r})$, in terms of this series is simply,

$$\varphi(\underline{r}) = \frac{G}{r}\int_V \rho(\underline{r}') \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\gamma)\, dV \tag{2.7}$$

but

$$\cos\gamma = \cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'\cos(\theta - \theta') \tag{2.8}$$

so that $P_n(\cos\gamma)$ may be expanded as a function of φ, φ', θ, and θ' by the addition theorem for spherical harmonics (commonly known as the decomposition formula).

$$P_n(\cos\gamma) = P_n(\cos\varphi)\, P_n(\cos\varphi')$$

$$+ 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} \cos m(\theta - \theta') P_{nm}(\cos\varphi)\, P_{nm}(\cos\varphi') \tag{2.9}$$

where the P_{nm}'s are called associated Legendre functions and may be calculated as

$$P_{nm}(\nu) = (1 - \nu^2)^{m/2} \frac{d^m}{d\nu^m} P_n(\nu) \tag{2.10}$$

and where

$$\cos m(\theta - \theta') = \cos m\theta\cos m\theta' + \sin m\theta\sin m\theta' \tag{2.11}$$

Before substituting for $P_n(\cos\gamma)$, the first few terms of Eq. (2.7) may be separated as shown below.

$$\varphi(\underline{r}) = \frac{G}{r} \int_V \rho(\underline{r}') \, dV$$

$$+ \frac{G}{r^2} \int_V r' \cos\gamma \, \rho(\underline{r}') \, dV$$

$$+ \frac{G}{r} \int_V \rho(\underline{r}') \sum_{n=2}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\gamma) \, dV \qquad (2.12)$$

The first term in Eq. (2.12) integrates as

$$\frac{G}{r} \int_V \rho(\underline{r}') \, dV = \frac{Gm}{r}$$

where m is the total mass of the body. The second term is identically zero, if the origin of the coordinate system is taken to be at the center of mass of the body. Hence,

$$\varphi(\underline{r}) = \frac{Gm}{r} [1 + \int_V \rho(\underline{r}') \sum_{n=2}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\gamma) \, dV] \qquad (2.13)$$

Substituting Eq. (2.9) into Eq. (2.13) yields

$$\varphi(\underline{r}) = \frac{Gm}{r} [1 + \sum_{n=2}^{\infty} \left(\frac{r_e}{r}\right)^n P_n(\cos\varphi) \int_V \left(\frac{r'}{r_e}\right)^n P_n(\cos\varphi') \rho(\underline{r}') \, dV$$

$$+ 2 \sum_{n=2}^{\infty} \left(\frac{r_e}{r}\right)^n \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} \cos m\theta \, P_{nm}(\cos\varphi) \int_V \left(\frac{r'}{r_e}\right)^n P_{nm}(\cos\varphi') \cdot$$

$$\cos m\theta \, \rho(\underline{r}') \, dV$$

$$+ 2 \sum_{n=2}^{\infty} \left(\frac{r_e}{r}\right)^n \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} \sin m\theta \; P_{nm}(\cos \phi) \int_V \left(\frac{r'}{r_e}\right)^n P_{nm}(\cos \phi') \cdot$$

$$\sin m\theta' \; \rho(\underline{r}')dV] \tag{2.14}$$

where r_e is the given radius of the mass. Combining like coefficients of $P_{nm}(\cos \phi)$ reduces Eq. (2.14) to

$$\varphi(\underline{r}) = \frac{Gm}{r}[1 + \sum_{n=2}^{\infty} \left(\frac{r_e}{r}\right)^n P_n(\cos \phi) \int_V \left(\frac{r'}{r_e}\right)^n P_n(\cos \phi') \; \rho(\underline{r}') \, dV$$

$$+ 2 \sum_{n=2}^{\infty} \left(\frac{r_e}{r}\right)^n \sum_{m=1}^{n} [\frac{(n-m)!}{(n+m)!} P_{nm}(\cos \phi) \cos m\theta \int_V \left(\frac{r'}{r_e}\right)^n P_{nm}(\cos \phi') \cos m\theta' \cdot \rho(\underline{r}')dV$$

$$+ \frac{(n-m)!}{(n+m)!} P_{nm}(\cos \phi) \sin m\theta \int_V \left(\frac{r'}{r_e}\right)^n P_{nm}(\cos \phi') \sin m\theta' \; \rho(\underline{r}')dV] \tag{2.15}$$

By defining the following coefficients

$$J_n = - \int_V \left(\frac{r'}{r_e}\right)^n P_n(\cos \phi') \; \rho(\underline{r}')dV \tag{2.16}$$

$$C_{nm} = 2 \frac{(n-m)!}{(n+m)!} \int_V \left(\frac{r'}{r_e}\right)^n P_{nm}(\cos \phi') \cos m\theta' \; \rho(\underline{r}')dV \tag{2.17}$$

$$S_{nm} = 2 \frac{(n-m)!}{(n+m)!} \int_V \left(\frac{r'}{r_e}\right)^n P_{nm}(\cos \phi') \sin m\theta' \; \rho(\underline{r}')dV \tag{2.18}$$

The gravitational potential $\varphi(\underline{r})$ may be reduced to

$$\varphi(\underline{r}) = \frac{Gm}{r} [1 - \sum_{n=2}^{\infty} J_n \left(\frac{r_e}{r}\right)^n P_n(\cos \varphi)$$

$$+ \sum_{n=2}^{\infty} \left(\frac{r_e}{r}\right)^n \sum_{m=1}^{n} (C_{nm} \cos m\theta + S_{nm} \sin m\theta) P_{nm}(\cos \varphi) \tag{2.19}$$

For m = 0, the general term S_{nm} as defined by Eq. (2.18) involves the definite integral of sin $m\theta'$ so that $S_{no} = 0$ for all n, however C_{no} as defined by Eq. (2.17) may be written as

$$C_{no} = \int_V \left(\frac{r'}{r_e}\right)^n P_{no}(\cos \varphi') \rho (\underline{r}')dV \tag{2.20}$$

Since

$$P_{nm}(\nu) = (1 - \nu^2)^{m/2} \frac{d^m}{d\nu^m} P_n(\nu) \tag{2.21}$$

$$P_{no}(\nu) = P_n(\nu) \tag{2.22}$$

Eq. (2.20) may then be written as

$$C_{no} = \int_V \left(\frac{r'}{r_e}\right)^n P_n(\cos \varphi') \rho (\underline{r}')dV \tag{2.23}$$

and, by comparison with Eq. (2.16), it may be seen that

$$J_n = -C_{nm} \qquad \text{for} \quad m = 0 \qquad (2.24)$$

The general expression for the gravitational potential, Eq. (2.19) may then be condensed to

$$\varphi(\underline{r}) = \frac{Gm}{r}\left[1 + \sum_{n=2}^{\infty}\left(\frac{r_e}{r}\right)^n \sum_{m=0}^{n} (C_{nm}\cos m\theta + S_{nm}\sin m\theta)P_{nm}(\cos\varphi)\right] \qquad (2.25)$$

The terms in φ corresponding to $m = 0$ are called the zonal harmonics while those corresponding to $m = 0$ are called the tesseral harmonics. If it is desired to calculate the potential at any point around the earth, for example, $\varphi(\underline{r})$ is usually written with θ as the geographic longitude λ, φ as the geographic co-latitude $\pi/2 - L$ as shown below:

$$\varphi(\underline{r}) = \frac{Gm}{r}\left[1 + \sum_{n=2}^{\infty}\left(\frac{r_e}{r}\right)^n \sum_{m=0}^{n} (C_{nm}\cos m\lambda + S_{nm}\sin n\lambda)P_{nm}(\sin L)\right) \qquad (2.26)$$

In general, it is possible to calculate the force on a unit mass at $\underline{r}$ due to the mass distribution $\rho(\underline{r}')$ by taking the gradient of $\varphi(\underline{r})$ as expressed in Eq. (2.25).

$$\underline{f}(\underline{r}) = \underline{\nabla}\varphi(\underline{r}) \qquad (2.27)$$

The gradient of $\varphi(\underline{r})$ in spherical coordinates is

$$\underline{\nabla}\varphi(\underline{r}) = \frac{\partial\varphi}{\partial r}\underline{i}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\varphi}\underline{i}_\varphi + \frac{1}{r\sin\varphi}\frac{\partial\varphi}{\partial\varphi}\underline{i}_\theta \qquad (2.28)$$

so that

$$\underline{f}(\underline{r}) = -\frac{Gm}{r^2}\left[1 + \sum_{n=2}^{\infty}(n+1)\left(\frac{r_e}{r}\right)^n \sum_{m=0}^{n}(C_{nm}\cos m\theta +\right.$$

$$\left. S_{nm}\sin m\theta)\, P_{nm}(\cos\varphi)\right]\underline{i}_r$$

$$-\frac{Gm}{r^2}\left[\sum_{n=2}^{\infty}\left(\frac{r_e}{r}\right)^n \sum_{m=0}^{\infty}(C_{nm}\cos m\theta +\right.$$

$$\left. S_{nm}\sin n\lambda)\, P'_{nm}(\cos\varphi)\sin\varphi\right]\underline{i}_\varphi$$

$$+\frac{Gm}{r^2\sin\varphi}\left[\sum_{n=2}^{\infty}\left(\frac{r_e}{r}\right)^n \sum_{m=0}^{n}(S_{mn}\cos m\theta -\right.$$

$$\left. C_{mn}\sin m\theta)\, mP_{nm}(\cos\varphi)\right]\underline{i}_\theta \qquad (2.29)$$

The equation for this expansion of the gravitational potential of a distributed mass $\rho(\underline{r})$ in a region outside the mass is in essence a solution of Laplace's Equation:

$$\nabla^2 \varphi(\underline{r}) = \rho(\underline{r}) = 0 \qquad (2.30)$$

The solution to Eq. (2.30), $\varphi(\underline{r})$, is a harmonic function and, of course, is valid only where Eq. (2.30) holds, which is everywhere outside the attracting body. This solution was obtained as a power series expansion of $\varphi(\underline{r})$ in powers of $(1/r)$ as shown by Eq. (2.25), hence the convergence is fastest when r is large and slowest when r is small. Inside the mass $\rho(\underline{r})$ the solution is usually divergent

(Heiskanen and Moritz). On the surface, this solution is either very slow to converge or, in theory, must be considered divergent because of the variations in $\rho(\underline{r})$ beneath the surface. Therefore, an expansion of the potential in spherical harmonics is useful for high altitude orbital mechanics, but deteriorates for low altitudes due to the intrinsic nature of the expansion. Also, one would intuitively expect that a lot of information (terms in the expansion) is necessary to describe a rapidly varying fine structure such as the gravitational potential near the surface of the earth. The effect of the rapidly varying fine structure falls off compared to the low order terms in the expansion at high altitudes, again due to the $(\frac{1}{r})^n$ dependence.

2.2 Orbit Calculation Using Spherical Harmonics

The ideal two-body equation of motion which governs the motion of one body with respect to another is

$$\frac{d^2\underline{r}}{dt^2} = -\frac{\mu}{r^3}\underline{r} \qquad (2.31)$$

where $\underline{r}$ is the position vector of the secondary body with respect to the primary, or attracting body, and $\mu = G(m_1 + m_2)$; where G is the universal gravitational constant, and m_1 and m_2 are the masses of the primary and secondary body respectively. In practice, if $m_2 \ll m_1$, then Eq. (2.31) is usually written

$$\frac{d^2\underline{r}}{dt^2} = -\frac{Gm}{r^3}\underline{r} \qquad (2.32)$$

where $\underline{r}$ is the position vector from the distributed mass to the satellite or spacecraft, and m is just the total mass of the attracting body. The motion of a spacecraft near a distributed mass is not simple two-body motion due to a number of reasons. One of the most important disturbing accelerations is due to gravity anomalies, which shall be defined as the components of a gravitational field of a distributed mass remaining after the central force term ($-\frac{Gm}{r^3}\underline{r}$) has been removed. Let $\underline{a}_d(\underline{r})$ represent the disturbing acceleration as a function of $\underline{r}$, then

$$\frac{d^2\underline{r}}{dt^2} = -\frac{Gm}{r^3}\underline{r} + \underline{a}_d(\underline{r}) \tag{2.33}$$

so that

$$-\frac{Gm}{r^3}\underline{r} + \underline{a}_d(\underline{r}) = \underline{\nabla}\varphi(\underline{r}) \tag{2.34}$$

hence, it may be seen from comparison with Eq. (2.25) that the nominal potential $\varphi_{nom}(\underline{r})$ and the disturbing potential $\varphi_d(\underline{r})$ respectively are

$$\varphi_{nom}(\underline{r}) = \frac{Gm}{r} \tag{2.35}$$

and

$$\varphi_d(\underline{r}) = \frac{Gm}{r}\left[\sum_{n=2}^{\infty}\left(\frac{r_e}{r}\right)^n \sum_{m=0}^{n} (C_{nm}\cos m\theta + S_{nm}\sin m\theta)\, P_{nm}(\cos\phi)\right] \tag{2.36}$$

so that

$$\underline{a}_d(\underline{r}) = \underline{\nabla}\,\varphi_d(\underline{r}) \tag{2.37}$$

For purposes of computing an orbit given the initial conditions on $\underline{r}$ and $\underline{v}$ at $t = t_0$, $\underline{r}(t)$ may be expressed as (Battin, 1964)

$$\underline{r}(t) = \underline{r}_{osc}(t) + \underline{\delta}(t) \tag{2.38}$$

where $\underline{r}_{osc}(t)$ is the solution to the nominal two-body problem

$$\frac{d^2 \underline{r}_{osc}(t)}{dt^2} = - \frac{Gm}{r^3_{osc}(t)} \underline{r}_{osc}(t) \tag{2.39}$$

with $\underline{r}_{osc}(t_0) = \underline{r}(t_0)$, and $\underline{v}_{osc}(t_0) = \underline{v}(t_0)$

The differential equation governing $\underline{\delta}(t)$ may be derived by substituting Eq. (2.38) into Eq. (2.33) as below

$$\frac{d^2 \underline{r}_{osc}(t)}{dt^2} + \frac{d^2 \underline{\delta}(t)}{dt^2} = - \frac{Gm}{\underline{r}^3(t)} \underline{r}(t) + \underline{a}_d(t) \tag{2.40}$$

rearranging terms,

$$\frac{d^2 \underline{\delta}(t)}{dt^2} = + \frac{Gm}{r^3_{osc}(t)} \underline{r}_{osc}(t) - \frac{Gm}{r^3(t)} \underline{r}(t) + \underline{a}_d(t) \tag{2.41}$$

so that

$$\frac{d^2 \delta(t)}{dt^2} = \frac{Gm}{r^3_{osc}(t)} [(1 - \frac{r^3_{osc}(t)}{r^3(t)}) \underline{r}(t) - \underline{\delta}(t)] + \underline{a}_d(t) \tag{2.42}$$

with initial conditions $\underline{\delta}(t_0) = 0$ and $\dot{\underline{\delta}}(t_0) = 0$

It is convenient to integrate $\frac{d^2 \underline{r}_{osc}(t)}{dt^2}$ and $\frac{d^2 \underline{\delta}(t)}{dt^2}$ separately and add the results to obtain $\underline{r}(t)$ because the first integration yields the exact solution of the classical two-body problem which is well known, while the second integration produces a small correction to $\underline{r}_{osc}(t)$, $\underline{\delta}(t)$, provided of course, that $\underline{a}_d$ is relatively small. Numerical difficulties in evaluating $[1 - r_{osc}^3(t)/r^3(t)]$ of Eq. (2.42) may be avoided (Battin, 1964). By defining

$$1 - \frac{r_{osc}^3}{r^3} = - f(q) \tag{2.43}$$

and

$$f(q) = q \frac{3 + 3q + q^2}{1 + (1 + q)^{3/2}} \tag{2.44}$$

where

$$q = \frac{(\underline{\delta} - 2\underline{r})\cdot\underline{\delta}}{r^2} \tag{2.45}$$

Hence Eqs. (2.39) and (2.42) may be simultaneously solved to produce $\underline{r}(t)$ and $\underline{v}(t)$ given initial conditions $\underline{r}(t_0)$ and $\underline{v}(t_0)$ where $\underline{a}_d$ is a small disturbing acceleration. Eq. (2.42) must be solved by integration whereas Eq. (2.39) may either be integrated directly or solved by a numerical iteration algorithm.

APPENDIX A

The gravitational potential $\varphi(\underline{r})$ may be expressed in terms of the normalized coefficients $\overline{C}_{mn}$ and $\overline{S}_{mn}$ as shown below

$$\varphi(\underline{r}) = \frac{Gm}{r}\left[1 + \sum_{n=2}^{\infty}\left(\frac{r_e}{r}\right)^n \sum_{m=0}^{n} (\overline{C}_{nm} \cos m\lambda\right.$$

$$\left. + \overline{S}_{nm} \sin m\lambda)\, \overline{P}_{nm}(\cos \phi)\right] \qquad \text{(A-1)}$$

The normalized coefficients $\overline{C}_{nm}$ and $\overline{S}_{nm}$ are related to C_{nm} and S_{nm} by

$$C_{no} = \sqrt{2n+1}\;\overline{C}_{no} \qquad (m = 0)$$

and (A-2)

$$\begin{Bmatrix} C_{nm} \\ S_{nm} \end{Bmatrix} = \sqrt{\frac{2(2n+1)(n-m)!}{(n+m)!}} \begin{Bmatrix} \overline{C}_{nm} \\ \overline{S}_{nm} \end{Bmatrix} \quad (m \neq 0)$$

The coefficients $\overline{C}_{nm}$ and $\overline{S}_{nm}$ result from normalizing P_{nm} according to the integral of P_{nm}^2 over the unit sphere:

$$\frac{1}{4\pi}\int_{\text{unit sphere}} [P_{no}(\cos \phi)]^2 dS = \frac{1}{2n+1} \quad (m = 0)$$

and

$$\frac{1}{4\pi}\int_{\text{unit sphere}} \left[P_{nm}(\cos \phi)\begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix}\right]^2 dS = \frac{1}{2(2n+1)}\frac{(n+m)!}{(n-m)!}$$

$$(m \neq 0) \qquad \text{(A-3)}$$

hence, the integrals of (A-3) may be unity if

$$\overline{P}_{no} = \sqrt{(2n + 1)}\; P_{no}$$

and

$$\overline{P}_{nm} = \sqrt{\frac{2(2n + 1)(n - m)!}{(n + m)!}}\; P_{nm} \tag{A-4}$$

It may be easily seen that normalized coefficients as expressed in Eq. (A-2) must follow if $\varphi(\underline{r})$ expressed as a function of C_{nm}, S_{nm} and P_{nm} equals $\varphi(\underline{r})$ expressed as a function of $\overline{C}_{nm}$, $\overline{S}_{nm}$ and $\overline{P}_{nm}$.

For purposes of completeness, it will be noted that in the case of the earth, another not uncommon representation of the gravitational potential (Heiskanen and Moritz, 1967) is

$$\varphi(\underline{r}) = \frac{Gm}{r}\left\{1 - \sum_{n=2}^{\infty} \left(\frac{r_e}{r}\right)^n \sum_{m=0}^{n} \left[J_{nm} \cos m\theta + K_{nm} \sin m\theta \right] P_{nm}(\cos \varphi)\right\} \tag{A-5}$$

For this representation, it may easily be seen from comparison with Eq. (2.26) that

$$J_{nm} = - C_{nm} \tag{A-6}$$

and

$$K_{nm} = - S_{nm} \tag{A-7}$$

APPENDIX B

The values of J_{no} for the earth used in this thesis, $2 \leq n \leq 20$ (Lundquist, 1967, P. 52) in units of 10^{-6} are

J_2 = 1082.639, J_3 = -2.565, J_4 = -1.608, J_5 = -0.174,

J_6 = 0.542, J_7 = -0.419 J_8 = -0.128, J_9 = -0.022,

J_{10} = -0.338, J_{11} = 0.176, J_{12} = 0.053, J_{13} = -0.146,

J_{14} = -0.174, J_{15} = -0.065, J_{16} = 0.449, J_{17} = -0.052,

J_{18} = -0.324, J_{19} = -0.075, J_{20} = 0.334.

and the values of the normalized coefficients $\overline{C}_{nm}$ and $\overline{S}_{nm}$ in units of 10^{-6} for $2 \leq n \leq 15$, $1 \leq$ m 15 are (Lundquist, 1967, p. 63 and 64).

n	m	C_{nm}	S_{nm}	n	m	C_{nm}	S_{nm}
2	2	2.38	-1.35	10	1	0.10	-0.07
3	1	1.71	0.23	10	2	-0.08	-0.06
3	2	0.84	-0.51	10	3	-0.08	-0.05
3	3	0.66	1.43	10	4	-0.06	-0.08
4	1	-0.47	-0.39	10	5	0.02	-0.02
4	2	0.35	0.48	10	6	-0.04	-0.01
4	3	0.92	-0.24	10	7	0.04	-0.05
4	4	0.04	0.30	10	8	0.04	-0.05
5	1	-0.06	-0.05	10	9	0.05	-0.04
5	2	0.53	-0.21	10	10	0.03	-0.02
5	3	-0.40	0.07	11	1	-0.03	0.02

n	m	C_{nm}	S_{nm}	n	m	C_{nm}	S_{nm}
5	4	-0.20	0.02	11	2	0.05	-0.05
5	5	0.18	-0.56	11	3	0.01	-0.08
6	1	-0.08	0.01	11	4	-0.03	0.00
6	2	0.01	-0.27	11	5	0.03	0.02
6	3	-0.04	0.03	11	6	-0.03	-0.02
6	4	-0.08	-0.48	11	7	0.03	-0.03
6	5	-0.26	-0.46	11	8	0.04	-0.02
6	6	-0.02	-0.16	11	9	0.03	0.01
7	1	0.17	0.11	11	10	-0.03	-0.01
7	2	0.32	0.16	11	11	0.10	0.06
7	3	0.18	0.00	12	1	-0.09	-0.07
7	4	-0.16	-0.04	12	2	-0.06	0.02
7	5	0.07	-0.01	12	3	0.03	0.02
7	6	-0.23	0.10	12	4	-0.05	0.01
7	7	0.07	0.06	12	5	0.02	0.01
8	1	-0.01	-0.01	12	6	-0.01	0.01
8	2	0.04	0.04	12	7	-0.04	-0.02
8	3	-0.03	0.00	12	8	0.00	0.01
8	4	-0.17	-0.02	12	9	-0.01	0.02
8	5	-0.09	0.09	12	10	-0.01	0.00
8	6	-0.01	0.30	12	11	-0.05	-0.02
8	7	0.02	0.04	12	12	-0.01	-0.01
8	8	-0.18	0.03	13	1	0.00	0.04
9	1	0.11	0.00	13	2	-0.03	0.01
9	2	0.03	0.05	13	3	0.00	0.03
9	3	-0.03	-0.01	13	4	-0.01	-0.02
9	4	0.07	0.02	13	5	0.03	-0.02
9	5	-0.04	0.04	13	6	-0.03	0.05
9	6	0.04	0.01	13	7	-0.02	0.00
9	7	0.04	-0.02	13	8	-0.02	-0.01
9	8	0.13	0.00	13	9	0.02	0.05
9	9	0.08	0.04	13	10	0.04	-0.02

n	m	C_{nm}	S_{nm}
13	11	-0.02	0.01
13	12	-0.02	0.06
13	13	-0.07	0.00
14	1	-0.01	0.02
14	2	-0.01	-0.04
14	3	0.06	-0.03
14	4	0.00	0.00
14	5	0.05	-0.03
14	6	0.01	-0.03
14	7	0.03	0.02
14	8	-0.03	-0.03
14	9	0.03	0.07
14	10	0.04	0.01
14	11	0.04	0.01
14	12	0.05	-0.03
14	13	0.01	0.04
14	14	-0.04	0.02
15	1	0.01	-0.01
15	2	-0.02	-0.03
15	3	0.02	0.03
15	4	0.00	0.01
15	5	0.03	-0.02
15	6	0.03	-0.05
15	7	0.03	0.04
15	8	-0.06	0.00
15	9	0.00	0.04
15	10	0.02	0.01
15	11	0.01	0.01
15	12	-0.07	0.05
15	13	-0.05	-0.03
15	14	0.01	-0.03
15	15	-0.02	-0.01

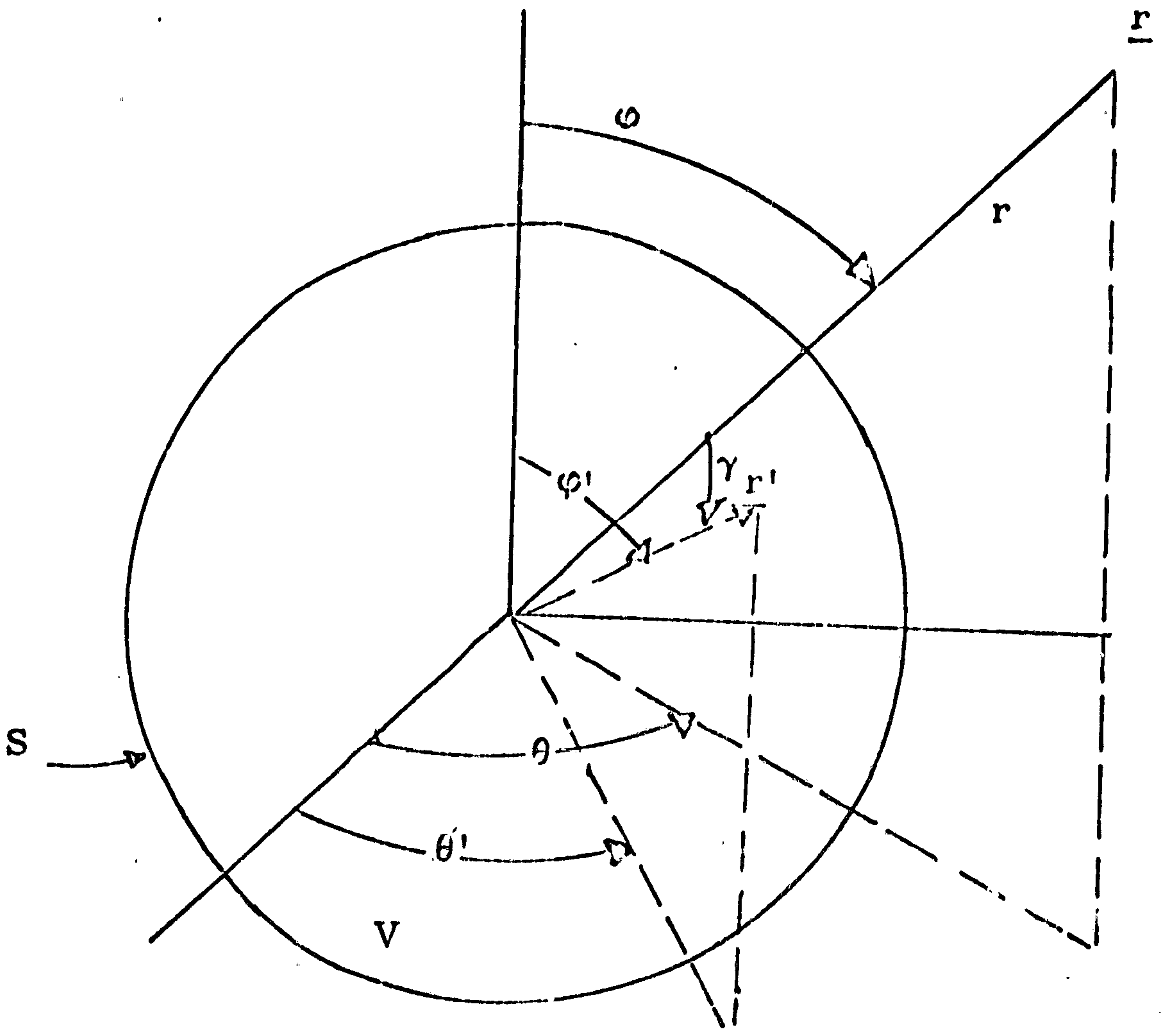

Figure 2.1 Coordinate System Definition

References

Battin, R. H. (1964). Astronautical Guidance, McGraw-Hill, Inc., New York.

Heiskanen, W. A. and Moritz, H. (1967). Physical Geodesy, W. H. Freeman and Co., San Francisco and London.

Laning, J. H. (1969). High Altitude Gravity Formulations, private communication.

Lundquist, C. A. (1967) Geodetic Satellite Results During 1967, Smithsonian Astrophysical Observatory, Cambridge, Mass., Special Report 264.